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WILSON NUMBER IN MIN-MAX MATRICES

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Abstract

Let T be a finite number of multiple set of real numbers taken as increasing order of numbers. The purpose of this article is to study the different properties of MIN matrix and MAX matrix of the set T with min (x_i, x_j) and max (x_i, x_j) as their (i, j) entries, respectively. We are going to do this by interpreting these matrices as Wilson meet and join matrices and applying the determinant formulae and the inverse formulae for Wilson MIN matrices and Wilson MAX matrices.

Keywords: MIN matrix, MAX matrix, meet matrix, join matrix, Wilson min matrix and Wilson max matrix.

1 Introduction

MIN and MAX matrices are simple-structured matrices that appear in many contexts in mathematics and statistics. As is pointed out in the next section, in some cases MIN matrices have an interpretation as covariance matrices of certain stochastic processes. Bhatia [1] shows that the MIN matrix [min (i, j)] is infinitely divisible, and in [2] he gives a more comprehensive treatment to this subject. Moyé studies the covariance matrix of Brownian motion, which appears to be a certain MIN matrix. Motivated by Moyé's work, Neudecker, Trenkler and Liu [3] defined a more general matrix

$$A = \begin{bmatrix} a_1 & a_1 & a_1 & & a_1 \\ a_1 & a_2 & a_2 & & a_2 \\ a_1 & a_2 & a_3 & & a_3 \\ & & \vdots & & \\ & & \vdots & & \\ & & \vdots & & \\ a_1 & a_2 & a_3 & & a_n \end{bmatrix}$$

(a_i are real numbers for all i = 1, . . . , n), and proposed the following problems:

– find a necessary and sufficient condition for A to be positive definite;
– find the determinant of A;
– find the inverse of A when A is nonsingular.

Two years later Chu, Puntanen and Styan made use of elementary matrix methods and provided answers to the above questions. Also in the field of pure mathematics MIN and MAX matrices have appeared in many contexts and by many authors. Probably the first such appearance can be found in the famous book by [4] Pólya and Szegö, where the reader is asked to calculate the determinant of the MIN matrix [min (i, j)] and also the determinants of some of its generalizations (in fact, all these exercises can be found already in the original German version of the book published in 1925). Meet matrices were defined by Rajarama Bhat [5] for the first time and in this same article MIN matrices are considered as an example. Da Fonseca studies the eigenvalues of certain MIN and MAX matrices via their matrix inverses, and in bounds for the values of trigonometric functions are found by underestimating the smallest eigenvalue of a MIN matrix. Also the connection between generalized Fibonacci numbers and the characteristic polynomials of MIN and MAX matrices have been studied recently.

2 Preliminaries

Let T = $\{x_1, x_2, x_3, \dots\dots\dots, x_n\}$ be a finite multiple set of real numbers, where $x_1 \le x_2 \le \dots \le x_n$ (in some cases, however, we need to assume that $x_1 < x_2 < \cdots < x_n$). The MIN matrix $(T)_{\min}$ of the set T has min (x_i, x_j) as its (i, j) entry, whereas the MAX matrix of the set T has max (x_i, x_j) as its (i, j) entry and is denoted by $[T]_{max}$. Both matrices are clearly square and symmetric and they may be written explicitly as

$$(T)_{min} = \begin{bmatrix} x_1 & x_1 & x_1 & & x_1 \\ x_1 & x_2 & x_2 & & x_2 \\ x_1 & x_2 & x_3 & & x_3 \\ & & \vdots & & \\ & & \vdots & & \\ & & \vdots & & \\ x_1 & x_2 & x_3 & & x_n \end{bmatrix} \text{ and } \quad [T]_{max} = \begin{bmatrix} x_1 & x_2 & x_3 & & x_n \\ x_2 & x_2 & x_3 & & x_n \\ x_3 & x_3 & x_3 & & x_n \\ & & \vdots & & \\ & & \vdots & & \\ & & \vdots & & \\ x_n & x_n & x_n & & x_n \end{bmatrix}$$

2.1 Remark

A partially ordered set (poset) is a pair $(P, \preceq)$, where P is a nonempty set and $\preceq$ is a reflexive, antisymmetric and transitive relation. A closed interval [x, y] in P is the set [6],

$$[x, y] = \{z \in P / x \preceq z \preceq y\}, \qquad x, y \in P.$$

Poset $(P, \preceq)$ is said to be locally finite if the interval [x, y] is finite for all x, y ∈ P. Poset $(P, \preceq)$ is a chain if $x \preceq y$ or $y \preceq x$ for all x, y ∈ P. A lattice is a poset, where the infimum $x \wedge y$ and the supremum $x \vee y$ exist for all x, y ∈ P. It is easy to see that every chain is a lattice with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

3 Some important results for meet and join matrices

In our study of MIN and MAX matrices we are going to make use of a couple of known results for meet and join matrices. The first one is about the structure of $(S)_f$. For any two subsets $S = \{x_1, x_2, \ldots \ldots \ldots x_n\}$ and $T = \{y_1, y_2, \ldots \ldots \ldots y_m\}$ of P, let [7] $E(S,T) = (e_{ij})$ denote the nxm incidence matrix defined as

$$e_{ij} = \begin{cases} 1 & \text{if } y_j \preccurlyeq x_i \\ 0 & \text{otherwise.} \end{cases}$$

Result 3.1

If S is meet closed [8], then

$$\det(S)_f = \prod_{v=1}^{n} \psi_{S,f}(x_v) = \prod_{v=1}^{n} \sum_{\substack{z \preccurlyeq x_v \\ z \npreceq x_t \\ t < v}} \sum_{w \preccurlyeq z} f(w)\mu_P(w, z).$$

Result 3.2

Suppose that S is meet closed [8]. If $(S)_f$ is invertible, then the inverse of $(S)_f$ is the n × n matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\psi_{S,f}(x_v)} \det E(S_i^k) \det E(S_j^k)$$

Where $E(S_i^k)$ is the $(n-1) \times (n-1)$ sub matrix of E(S) obtained by deleting the i^{th} row and the k^{th} column of E(S), or

$$b_{ij} = \sum_{x_i \vee x_j \preccurlyeq x_k} \frac{\mu_S(x_i, x_j)\mu_S(x_j, x_k)}{\psi_{S,f}(x_k)}$$

where μ_S is the Mobius function of the poset $(S, \preceq)$.

Result 3.3

If S is join closed [9], then

$$\det[S]_f = \prod_{v=1}^{n} \sum_{x_v < x_t} f(x_t)\mu_s(x_v, x_t) = \prod_{v=1}^{n} \sum_{\substack{x_v \preccurlyeq z \\ x_t \npreceq z \\ v<t}} \sum_{z \preccurlyeq w} f(w)\mu_P(z, w).$$

Result 3.4

Suppose that S is join closed [9]. If $[S]_f$ is invertible, then the inverse of $[S]_f$ is the n × n matrix $B = (b_{ij})$, where

$$b_{ij} = \sum_{k=1}^{n} \frac{(-1)^{i+j}}{\Phi_{S,f}(x_k)} \det E(S_k^i) \det E(S_k^j)$$

Where $E(S_i^k)$ is the $(n-1) \times (n-1)$ sub matrix of E(S) obtained by deleting the k^{th} row and the i^{th} column of E(S), or

$$b_{ij} = \sum_{x_k \preccurlyeq x_i \wedge x_j} \frac{\mu_S(x_k, x_i)\mu_S(x_k, x_j)}{\Phi_{S,f}(x_k)}$$

where μ_S is the Möbius function of the poset $(S, \preceq)$.

4 MIN and MAX matrices as meet and join matrices

The most straight forward attempt to interpret MIN and MAX matrices as meet and join matrices would be to set $(P, \preceq) = (R, \leq)$. This, however, cannot be done since the set of real numbers is not locally finite (meet and join matrices are usually studied via Möbius inversion, which requires the local finiteness property). Nevertheless, there is a way around the problem. We set $P = \{1, 2, \ldots, n\}$, $\preceq$ is the usual ordering $\leq$ of the integers and S = P. Since in this case $(P, \preceq)$ is a chain with n elements, it is trivially a locally finite lattice. Moreover, by defining $f : P \to R$ by $f(i) = z_i$ for all $i = 1, 2, \ldots, n$ we obtain $(S)_f = (T)_{min}$ and $[S]_f = [T]_{max}$.

Executing the Möbius inversion is now easy due to the simple chain-structure of the poset $(P, \preceq)$ (general information about Mobius inversion and Mobius functions on posets can be found). For the Möbius function of the chain $(P, \preceq)$ we have for $i, j \in P$ that

$$\mu_P(j,i) = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases}$$

The function μ_P can then be used to define two other functions ψ_P and Φ_P as

$$\psi_P(1) = x_1, \qquad \psi_P(i) = \sum_{1 \leq j \leq i} \mu_P(i,j)x_i = x_i - x_{i-1} \qquad for\ 1 < i \leq n.$$

$$\Phi_P(n) = x_n, \qquad \Phi_P(n) = \sum_{i \leq j \leq n} \mu_P(i,j)x_j = x_i - x_{i+1} \qquad for\ 1 < i \leq n.$$

It turns out that the values of the functions ψ_P and Φ_P characterize many key properties of the matrices $(T)_{min}$ and $[T]_{max}$ by [10].

5 SOME DEFINITIONS

5.1 MINIMUM AND MAXIMUM MATRICES

Let us define the matrix operations ∧ and ∨ by $(a_{i,j})_{nxm} \wedge (b_{i,j})_{nxm} = (\min(a_{i,j}, b_{i,j}))$ and $(a_{i,j})_{nxm} \vee (b_{i,j})_{nxm} = (\max(a_{i,j}, b_{i,j}))$. Let C denote the nxn matrix with $c_{i,j} = x_i$ for all $1 \leq i, j \leq n$. Then $(T)_{min} = C \wedge C^T$ and $[T]_{max} = C \vee C^T$.

5.2 WILSON MATRIX

Let $(P, \preceq)$ be a meet-semi lattice and Let S = $\{x_1, x_2, \ldots \ldots \ldots x_n\}$ be a set of distinct positive integers and a subset of P. Then S is an nxn matrix and $[M] = (m_{ij})$, where $m_{ij} = [(x_i, x_j) - 1]! + 1$, call it to be Wilson matrix on S [11].

5.3 WILSON MINIMUM MATRICES

Let S = $\{x_1, x_2, \ldots \ldots \ldots x_n\}$ be a set of distinct positive integers and the nxn matrix and $[M] = (m_{ij})$, where $m_{ij} = [\,min(x_i, x_j) - 1]! + 1$, , call it to be Wilson MIN matrix on S.

5.4 WILSON MAXIMUM MATRICES

Let S = $\{x_1, x_2, \ldots \ldots \ldots x_n\}$ be a set of distinct positive integers and the nxn matrix and $[M] = (m_{ij})$, where $m_{ij} = [\,max(x_i, x_j) - 1]! + 1$, , call it to be Wilson MAX matrix on S.

6 DETERMINANTS OF WILSON MIN AND WILSON MAX MATRICES

Theorem 6.1

We consider the determinants of the matrices,

$$\det (T)_{\min} = \psi_p(1)\psi_p(2) \ldots \ldots \ldots . \psi_p(n) = x_1(x_2 - x_1)(x_3 - x_2) \ldots \ldots \ldots (x_n - x_{n-1}).$$

$$\det [T]_{\max} = \Phi_p(1)\Phi_p(2) \ldots \ldots \ldots . \Phi_p(n) = (x_1 - x_2)(x_2 - x_3) \ldots \ldots \ldots (x_{n-1} - x_n)x_n.$$

Proof: These determinant formulas follow directly from Result 3.1 and Result 3.2.
.

Theorem 6.2

Next we consider the determinants of the Wilson min and max matrices

$$\text{Det Wil } (T)_{\min} = \psi_p(1)\psi_p(2) \ldots \ldots \ldots . \psi_p(n) = x_1(x_2 - x_1)(x_3 - x_2) \ldots \ldots \ldots (x_n - x_{n-1}),$$
$$\text{where } x_i = [\,min(x_i, x_j) - 1]! + 1 \quad j = 1,2, \ldots \ldots n.$$

$$\text{Det Wil } [T]_{\max} = \Phi_p(1)\Phi_p(2) \ldots \ldots \ldots . \Phi_p(n) = (x_1 - x_2)(x_2 - x_3) \ldots \ldots \ldots (x_{n-1} - x_n)x_n,$$
$$\text{where } x_i = [\,max(x_i, x_j) - 1]! + 1 \quad j = 1,2, \ldots \ldots n.$$

Example 1

If S = { 4, 5 } is a lower closed set. Consider 2x2 Wilson Min matrix on S is

$$\text{Wil } (S)_{\min} = \begin{bmatrix} 7 & 7 \\ 7 & 25 \end{bmatrix}$$

Det Wil $(S)_{\min} = x_1(x_2 - x_1) = 7(25 - 7) = 126$.

Consider 2x2 Wilson Max matrix on S is

$$\text{Wil } [S]_{\max} = \begin{bmatrix} 7 & 25 \\ 25 & 25 \end{bmatrix}$$

Det Wil $[S]_{\max} = (x_1 - x_2)x_1 = (7 - 25)25 = -450$.

Example 2

If S = {2, 3, 5} is a lower closed set.

Consider 3x3 Wilson Min matrix on S is

$$\text{Wil (S)}_{\text{min}} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 25 \end{bmatrix}$$

Det Wil (S) $_{\text{min}}$ = $x_1(x_2 - x_1)(x_3 - x_2)$ = 2x1x22 = 44.

Consider 3x3 Wilson Max matrix on S is

$$\text{Wil [S]}_{\text{max}} = \begin{bmatrix} 2 & 3 & 25 \\ 3 & 3 & 25 \\ 25 & 25 & 25 \end{bmatrix}$$

Det Wil [S] $_{\text{max}}$ = $(x_1 - x_2)(x_2 - x_3)x_3$ = (-1) x (-22) x25 = 550.

Example 3

If S = { 3, 4, 5 } is a lower closed set.

Consider 3x3 Wilson Min matrix on S is

$$\text{Wil (S)}_{\text{min}} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 7 & 7 \\ 3 & 7 & 25 \end{bmatrix}$$

Det Wil (S) $_{\text{min}}$ = $x_1(x_2 - x_1)(x_3 - x_2)$ = 3x4x18 = 216.

Consider 3x3 Wilson Max matrix on S is

$$\text{Wil [S]}_{\text{max}} = \begin{bmatrix} 3 & 7 & 25 \\ 7 & 7 & 25 \\ 25 & 25 & 25 \end{bmatrix}$$

Det Wil [S] $_{\text{max}}$ = $(x_1 - x_2)(x_2 - x_3)x_3$ = (-4) x (-18) x25 = 1800.

7 INVERSE OF WILSON MIN AND WILSON MAX MATRICES

Under the assumption that the elements of the set T are distinct the MIN and MAX matrices of the set T are usually invertible. Next we shall find their inverses.

Theorem 7.1:

[12] Suppose that the elements of the set T are distinct. If $x_1 \neq 0$, then the MIN matrix is invertiable and the inverse matrix is the nxn tridiagonal matrix B = (b_{ij}), where

$$B_{ij} = \begin{cases} 0 & if\ |i-j| > 1 \\ \frac{x_2}{x_1(x_2 - x_1)} & if\ i = j = 1 \\ \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i} & if\ 1 < i = j < n \\ \frac{1}{x_n - x_{n-1}} & if\ i = j = n \\ \frac{-1}{x_i - x_j} & if\ |i-j| = 1. \end{cases}$$

where $x_i = [\, min(x_i, x_j) - 1]! + 1 \quad i = 1,2, \ldots\ldots n.$

Theorem 7.2:

If $x_n \neq 0$, then the inverse of the MAX matrix is invertible and the inverse matrix is the nxn tridiagonal matrix C = (C_{ij}), where

$$C_{ij} = \begin{cases} 0 & if\ |i-j| > 1 \\ \frac{1}{(x_1 - x_2)} & if\ i = j = 1 \\ \frac{1}{x_{i-1} - x_i} + \frac{1}{x_i - x_{i+1}} & if\ 1 < i = j < n \\ \frac{1}{x_{n-1} - x_n} + \frac{1}{x_n} & if\ i = j = n \\ \frac{1}{x_i - x_j} & if\ |i-j| = 1. \end{cases}$$

where $x_i = [\, max(x_i, x_j) - 1]! + 1 \quad i = 1,2, \ldots\ldots n.$

Proof:

The inverse formulas follow straight from Result 3.3 and Result 3.4. An elementary approach would be to construct the supposed inverse matrices and multiply them with the matrices $(T)_{min}$ and $[T]_{max}$.

Example 4

(S) is a Wilson Min matrix on lower closed set S = { 4, 5 }. Then by definition 6.1

$$\text{Wil } S_{min}^{-1} = \text{B} = (\text{b}_{ij})$$

Therefore since Wil S_{min}^{-1} = B is the symmetric we have

$$\text{Wil } S_{min}^{-1} = \text{B} = \begin{bmatrix} \frac{25}{126} & \frac{-1}{18} \\ \frac{-1}{18} & \frac{1}{18} \end{bmatrix}$$

[S] is a Wilson Max matrix on lower closed set S = { 4, 5 }. Then by definition 6.2

$$\text{Wil } S_{max}^{-1} = \text{C} = (\text{c}_{ij})$$

Therefore since Wil S_{max}^{-1} = C is the symmetric we have

$$\text{Wil } S_{max}^{-1} = \text{C} = \begin{bmatrix} \frac{-1}{18} & \frac{1}{18} \\ \frac{1}{18} & \frac{-7}{450} \end{bmatrix}$$

Example 5

(S) is a Wilson Min matrix on lower closed set S = {2, 3, 5}. Then by definition 6.1

$$\text{Wil } S_{min}^{-1} = \text{B} = (\text{b}_{ij})$$

Therefore since Wil S_{min}^{-1} = B is the symmetric tridiagonal we have

$$\text{Wil } S_{min}^{-1} = \mathrm{B} = \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ -1 & \frac{23}{22} & \frac{-1}{22} \\ 0 & \frac{-1}{22} & \frac{1}{22} \end{bmatrix}$$

[S] is a Wilson Max matrix on lower closed set S = {2, 3, 5}. Then by definition 6.2

$$\text{Wil } S_{max}^{-1} = \mathrm{C} = (\mathrm{c_{ij}})$$

Therefore since Wil $S_{max}^{-1} = \mathrm{C}$ is the symmetric tridiagonal we have

$$\text{Wil } S_{max}^{-1} = \mathrm{C} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & \frac{-23}{22} & \frac{1}{22} \\ 0 & \frac{1}{22} & \frac{-3}{550} \end{bmatrix}$$

Example 6

(S) is a Wilson Min matrix on lower closed set S = { 3, 4, 5 }. Then by definition 6.1

$$\text{Wil } S_{min}^{-1} = \mathrm{B} = (\mathrm{b_{ij}})$$

Therefore since Wil $S_{min}^{-1} = \mathrm{B}$ is the symmetric tridiagonal we have

$$\text{Wil } S_{min}^{-1} = \mathrm{B} = \begin{bmatrix} \frac{7}{12} & \frac{-1}{4} & 0 \\ \frac{-1}{4} & \frac{11}{36} & \frac{-1}{18} \\ 0 & \frac{-1}{18} & \frac{1}{18} \end{bmatrix}$$

[S] is a Wilson Max matrix on lower closed set S = { 3, 4, 5 }. Then by definition 6.2

$$\text{Wil } S_{max}^{-1} = \mathrm{C} = (\mathrm{c_{ij}})$$

Therefore since Wil $S_{max}^{-1} = \mathrm{C}$ is the symmetric tridiagonal we have

$$\text{Wil } S_{max}^{-1} = \mathrm{C} = \begin{bmatrix} \frac{-1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{-11}{36} & \frac{1}{18} \\ 0 & \frac{1}{18} & \frac{-7}{450} \end{bmatrix}$$

CONCLUSION:

In this paper, the different properties of MIN and MAX matrices of the set T with min (x_i, x_j) and max (x_i, x_j) as their (i, j) entries like determinant value and inverse of MIN and MAX matrices have been studied. The study is carried out by applying known results of meet and joins matrices to Wilson min and Wilson max matrices. In future the various matrices may be considered under this study.

7 REFERENCES:

1. R. Bhatia, *Infinitely divisible matrices*, Amer. Math. Monthly 113 no. 3, 221–235, 2006.
2. R. Bhatia, *Min matrices and mean matrices*, Math. Intelligencer 33 no. 2, 22–28, 2011.
3. H. Neudecker, G. Trenkler, and S. Liu, *Problem section*, Stat Papers 50, 221–223, 2009.

4. G. Pólya and G. Szegö, *Problems and Theorems in Analysis II*, Vol. II, 4th ed., Springer, 1971.
5. B.V. Rajarama Bhat, *On greatest common divisor matrices and their applications*, Linear Algebra Appl. 158, 77–97, 1991.
6. S. Puntanen, G. P. H. Styan, and J. Isotalo, *Matrix Tricks for Linear Statistical Models -Our Personal Top Twenty*, 1st ed., Springer, 2011.
7. P. Haukkanen, *On meet matrices on posets*, Linear Algebra Appl. 249, 111–123, 1996.
8. E. Altinisik, N. Tuglu, and P. Haukkanen, *Determinant and inverse of meet and join matrices*, Int. J. Math. Math. Sci. Article ID 37580 2007.
9. M. Mattila and P. Haukkanen, *Determinant and inverse of join matrices on two sets*, Linear Algebra Appl. 438, 3891–3904, 2013.
10. Mika mattila and Pentti Haukkanen, *Studying the various properties of MIN and MAX matrices-elementary vs. more advanced methods*, Spec. Matrices, 4:101-109, 2016.
11. Serife Buyukkose, *The mersenne meet matrices on posets*, Int. J. Contemp. Math. Sci., Vol. 1, no. 10, 469-474, 2006.
12. Dr. N. Elumalai, Mrs. R. Muthamizh Selvi, *Max- Min Matrices Using Mersenne Number*, JETIR1808455, Volume 5, Issue 8, 2018.